Solution to
Fourth International Mathematics Assessment for Schools
Round 1 of Junior Division

1. What is the value of $2014-1^{204}+\sqrt{(-2014)^2}$?

   (A) 1 (B) -1 (C) -2087

   (D) 4027 (E) 4029

【Suggested Solution】

Original Expression $=2014-1+2014=4027$. Hence, we select option (D).

Answer: (D)

2. A compass costs 15.40 dollars and a ruler costs 8.65 dollars. How many more dollars does the compass cost than the ruler?

   (A) 7.25 dollars (B) 7.75 dollars (C) 24.05 dollars

   (D) 6.25 dollars (E) 6.75 dollars

【Suggested Solution】

The cost of a compass is more expensive than the cost of a ruler by 15.4－8.65=6.75 dollars. Hence, we select option (E).

Answer: (E)

3. The two stars in the diagram represent the same number. The sum of the three numbers in the second row is equal to twice the sum of the three numbers in the first row. What number does each star represent?

5	6	☆		
		☆	19	20

   (A) 7 (B) 8 (C) 13 (D) 17 (E) 18

【Suggested Solution】

From the given information, it shows the sum of the three numbers in the second row is equal to twice the sum of the three numbers in the first row, it follows that the difference of the sum of three numbers in the second row and the sum of three numbers in the first row equal the sum of three numbers in the first row, that is; the difference of these two rows is 19 + 20－5－6 = 28, then we have ☆ = 28－5－6 = 17. Thus, we select option (D).

Answer: (D)

4. In a restaurant, one cup of tea and two cups of coffee cost 78 dollars, while three cups of tea and one cup of coffee cost 94 dollars. How many more dollars does a cup of coffee cost than a cup of tea?

   (A) 2 (B) 4 (C) 6 (D) 10 (E) 12

【Suggested Solution】

From the given information, we know that 6 cups of tea and 2 cups of coffee cost $94 \times 2 = 188$ dollars, so 5 cups of tea cost $188 - 78 = 110$ dollars, it follows that the cost of 1 cup of tea is 22 dollars, and cost of 1 cup of coffee is $94 - 22 \times 3 = 28$ dollars, thus 1 cup of coffee is more expensive than 1 cup of tea by $28 - 22 = 6$ dollars. Hence, we select option (C).

Answer: （C）

5. When two numbers are divided by 5, the respective remainders are 4 and 2. What is the remainder when the sum of the two numbers is divided by 5?

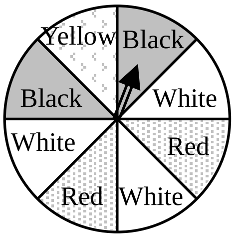
（A）0 （B）1 （C）2 （D）3 （E）4

【Suggested Solution】

The sum of the remainders of two numbers is 2 + 4 = 6, continue dividing this remainder by 5, we have the final remainder 1. Hence, we select option (B).

Answer: （B）

6. A circular spinner is divided into eight equal sectors. Two of them are painted red, two black, three white and one yellow. What is the probability for the pointer of the spinner to be pointing at a black sector?



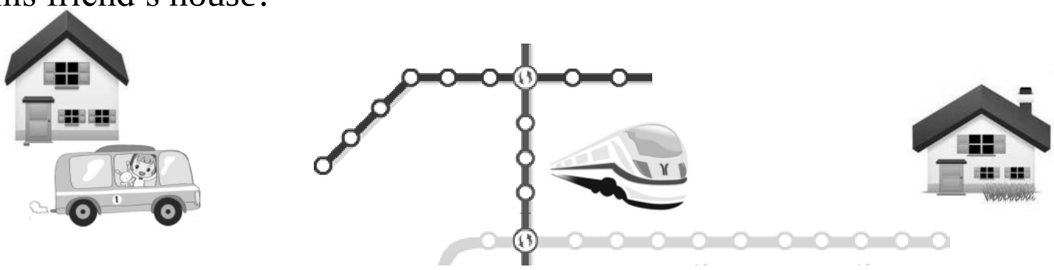
（A）$\frac{3}{4}$ （B）$\frac{1}{4}$ （C）$\frac{3}{8}$ （D）$\frac{1}{8}$ （E）$\frac{1}{2}$

【Suggested Solution】

In eight equal sectors of the circular spinner, 2 parts are black, so the probability for the pointer of the spinner to be pointing at a black sector is $\frac{2}{8} = \frac{1}{4}$. Hence, we select option (B).

Answer: （B）

7. To visit a friend, Rod must take the bus to the nearest Metro station, and this takes 15 minutes. He has to ride the Metro train for 20 stops, each taking 2.5 minutes. He also has to change trains twice, and it takes 3 minutes each time. Finally, after exiting the Metro, he still has to walk another 12 minutes before reaching his friend’s place. How many minutes does Rod have to spend traveling to his friend’s house?



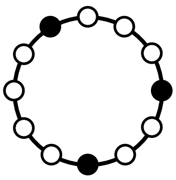
（A）55 （B）67 （C）80 （D）83 （E）90

【Suggested Solution】

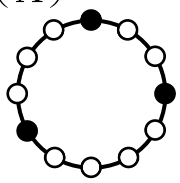
From the given information, Rod must spend a total of 15 + 2.5 × 20 + 3 × 2 + 12 = 83 minutes. Hence, we select option (D) .

Answer: (D)

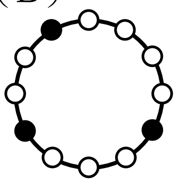
8. On a table there is a ring, there are 12 equally spaced beads on the ring, 3 of which are black, as shown in the diagram. Which of the following five figures cannot be obtained from the given figure by rotating the ring on the table?



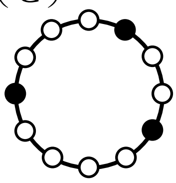
(A)



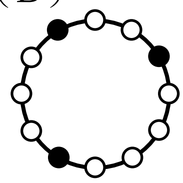
(B)



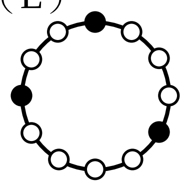
(C)



(D)



(E)



【Suggested Solution】

Observe the arrangement as regards the number of small white circles between two black circles of the given diagram in counterclockwise direction: four white small circles, two small white circles and then three small white circles. In all the five options, only option (E) does not comply with this kind of arrangement. Hence, we select (E).

Answer: (E)

9. If a, x and y are real numbers such that $|2y-12|+\sqrt{ax-y}=0$, what is the value of the product axy?

(A) 0 (B) 6 (C) 12

(D) 36 (E) impossible to determine

【Suggested Solution】

The two terms at the left side of the given expressions are both non-negative, it follows the values of both equal to 0, then $2y-12=0$ and $ax-y=0$, so we have $ax = y = 6$, this implies $axy = 36$. Hence, we select option (D).

Answer: (D)

10. How many integers a satisfies $|2a+7|+|2a-1|=8$?

(A) 9 (B) 8 (C) 5 (D) 4 (E) infinite

【Suggested Solution】

Represent the numbers on the left side of the given equation in a number line and interpret them, since the distance of two points whose coordinate is -7 and 1 is exactly 8, that means the coordinate $2a$ must between the two coordinates -7 and 1, so we have $-7\le 2a\le 1$, or it is equivalent as $-\frac{7}{2}\le a\le \frac{1}{2}$. Since a must be integers, then the possible values of a are -3, -2, -1 and 0. Therefore, there are 4 values of a that satisfy the given equation. Hence, we select option (D).

Answer: (D)

11. If a and b are prime numbers such that $a^2 - 7b - 4 = 0$, what is the value of $a+b$?

(A) 5 (B) 8 (C) 9 (D) 10 (E) 13

【Suggested Solution 1】

It is obvious that $a = 3$ has not met the condition of the problem, then a cannot be a multiple of 3. Thus, when a^2 is divided by 3 will give a remainder of 1. Because when 4 is divided by 3 will also give a remainder of 1, so that when $7b$ is divisible by 3, it follows b is also multiple of 3, that is; $b = 3$. Therefore, $a = 5$ and $a + b = 8$. Hence, we select option (B).

【Suggested Solution 2】

Rewrite the given equation as $(a-2)(a+2) = 7b$, obviously $a = 2, 3$ does not meet the condition of the given problem, so both factors of the left side must be greater than 1 and since the two factors in the right side are prime numbers, then the two factors of the left side are also prime numbers.

When $a-2=b$ and $a+2=7$, we have $a = 5$, $b = 3$ which meet the condition of the problem! So, $a + b = 8$.

When $a-2=7$ and $a+2=b$, we have $a = 9$, $b = 11$, does not meet the condition of the problem!

Therefore, $a + b = 8$. Hence, we select option (B).

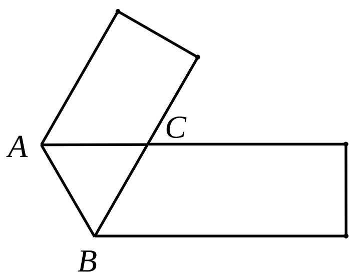
Answer: (B)

12. The diagram shows a strip of paper folded along the segment AB. If $\angle ACB = 60°$ and the area of triangle ABC is $\sqrt{3}\,\text{cm}^2$, what is the width, in cm, of this strip?

(A) 1 (B) $\sqrt{3}$

(C) $\frac{\sqrt{3}}{2}$ (D) $\frac{2\sqrt{3}}{3}$

(E) impossible to determine



【Suggested Solution】

By method of folding paper and Alternate Interior Angles Theorem, we have $\angle CAB = \angle CBA = \frac{180° - 60°}{2} = 60°$, so ΔABC is an equilateral triangle.

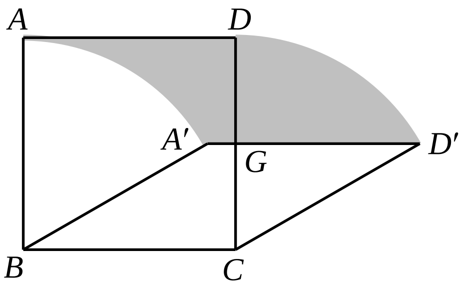
Let $AC = x$, then the altitude of ΔABC is $\frac{\sqrt{3}}{2}x$ and the area of ΔABC equals to $\frac{\sqrt{3}}{4}x^2$ cm^2, so that $\frac{\sqrt{3}}{4}x^2 = \sqrt{3}$, this implies $x = 2$.

Therefore, the width of the paper is $2 \times \frac{\sqrt{3}}{2} = \sqrt{3}\,\text{cm}$.

Hence, we select option (B).

Answer: (B)

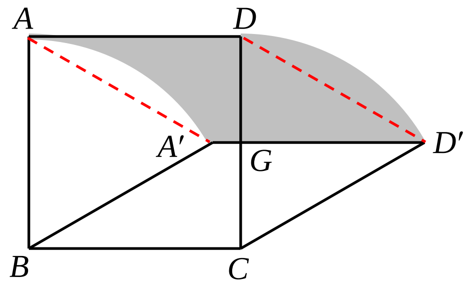
13. *ABCD* is a square of side length 10 cm. The segment *BC* is fixed. The segment *AD* moves in the plane to the segment *A′D′* so that the lengths *AB*, *DC* and *AD* do not change. What is the area, in cm^2, of the shaded region in the diagram when the segment *A′D′* intersects the segment *CD* at its midpoint *G*?



（A）50 （B）$\frac{50\pi}{3}$ （C）60 （D）100 （E）$\frac{100\pi}{3}$

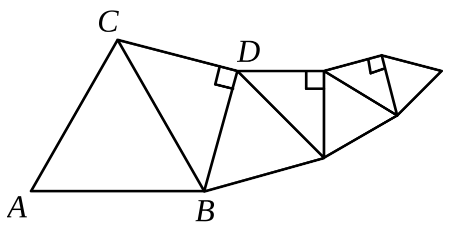
【Suggested Solution】

Connect *AA′* and *D′D*, Apply Cut and Paste method we know the area of the shaded region equals to the area of parallelogram *AA′D′D*, in parallelogram *AA′D′D* we have base *AD* = 10 cm, altitude *GD* = 5 cm and so, the area is 10 × 5 = 50 cm^2. Hence, we select option (A).



Answer:（A）

14. We start with an equilateral triangle *ABC* of area 80 cm^2. We construct a right isosceles triangle *BCD* using *BC* as the hypotenuse. Then we construct an equilateral triangle using *BD* as a side. This continued alternately, as shown in the diagram. What is the area, in cm^2, of the fourth equilateral triangle?



（A）1.25 （B）5 （C）6.4 （D）10 （E）40

【Suggested Solution】

Using Pythagoras' Theorem, each side length of an equilateral triangle is $\frac{\sqrt{2}}{2}$ times of the previous one, so the area of the recent equilateral triangle is $\frac{1}{2}$ that of the previous equilateral triangle. Thus, area of the fourth equilateral triangle in the given diagram is $80\times\left(\frac{1}{2}\right)^3=10\,\text{cm}^2$. Hence, we select option (D).

Answer:（D）

15. We wish to spend 100 dollars to buy 18 stamps, each costing 4 dollars, 8 dollars or 10 dollars. We must buy at least 1 stamp of each of the three kinds. How many different ways can the buying of stamps be possible?

（A）1 （B）2 （C）3 （D）4 （E）5

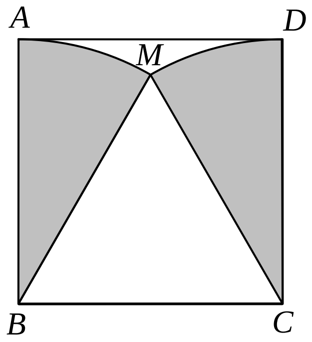
【Suggested Solution】

Let x represent the number of stamps that cost 8 dollars and y represent the number of stamps that cost 10 dollars, then $18-x-y$ represent the number of stamps that cost 4 dollars. From the given information, we obtain $8x+10y+4(18-x-y)=100$ and after simplifying we have $2x+3y=14$, notice that both x and y must be positive integers, then there are only two possible solution sets: $x = 4$, $y = 2$ and $x = 1$, $y = 4$. By trial and check, these two sets of solution meet the condition $18-x-y$ is a positive integer, so there are two different ways of buying the stamps. Hence, we select option (B).

Answer: (B)

16. The sectors *MAB* and *MCD* are inside the square *ABCD* of side length 10 cm, as shown in the diagram. What is the total area, in cm^2, of these two sectors, correct to 1 decimal place? Take π=3.14.

(A) 52.3 (B) 78.5 (C) 104.7 (D) 157.0 (E) 314.0



【Suggested Solution】

From the given information, $\triangle BCM$ is an equilateral triangle, then $\angle ABM=\angle MCD$ =30°.

Total area of the two sectors is $3.14\times 10^2\times\dfrac{30}{360}\times 2\approx 52.3\,\text{cm}^2$. Hence, option (A).

Answer: (A)

17. Three different positive integers *m, n* and *p* are such that

$$(m-3)(n-3)(p-3)=4.$$

What is the value of $m + n + p$?

(A) 5 (B) 6 (C) 8 (D) 14 (E) 15

【Suggested Solution】

From the given information, $m-3$, $n-3$, $p-3$ are three distinct integers such that their product is 4. From parity checking, maybe all the three integers are positive or one is positive integer and two negative integers. When all the three integers are positive, then the least product is $1 \times 2 \times 3 = 6$, does not meet the condition of the problem. When one of them is positive and the other two are negative, let us assume $m-3<0$, $n-3<0$, $p-3>0$, due to the fact that the positive integers less than 3 are 1 and 2, so that *m, n* must be equal to 1 and 2, and so $p = 5$. The sum of these three integers is 8. Hence, we select option (C).

Answer: (C)

18. If $x<y<0$ and $x^2+y^2=4xy$, what is the value of $\frac{x+y}{x-y}$?

(A) $\sqrt{3}$ (B) $-\sqrt{3}$ (C) 3 (D) $\sqrt{6}$ (E) $-\sqrt{6}$

【Suggested Solution】

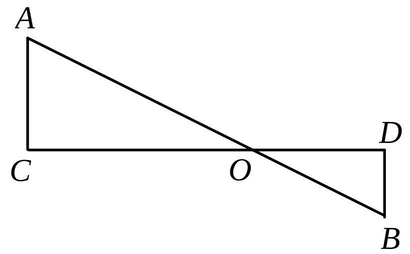
Since $x<y<0$ so we know that both $x+y$, $x-y$ are negative numbers,

it follows that $\frac{x+y}{x-y}>0$, and $\left(\frac{x+y}{x-y}\right)^2=\frac{x^2+y^2+2xy}{x^2+y^2-2xy}=\frac{6xy}{2xy}=3$.

Therefore $\frac{x+y}{x-y}=\sqrt{3}$ 。Hence, we select option (A).

Answer: (A)

19. The diagram shows two right triangles *OAC* and *OBD*. The lengths of three of the segments *AB*, *AC*, *CD* and *DB* are 12 cm, 6 cm and 3 cm. What is the number of possible lengths, in cm, of the fourth segment?



(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

【Suggested Solution】

AB is obvious the longest among the four line segments, then *AB* = 12 or *AB* = *x*:

(1) If *AB* = 12: When *CD* = *x*, $12^2=x^2+(3+6)^2$, $x=3\sqrt{7}$.

When *CD* = 6, $12^2=6^2+(3+x)^2$, $x=6\sqrt{3}-3$.

When *CD* = 3. $12^2=3^2+(6+x)^2$, $x=3\sqrt{15}-6$.

(2) If *AB* = *x*: When *CD* = 12, $x^2=12^2+(6+3)^2$, $x=15$.

When *CD* = 6. $x^2=6^2+(12+3)^2$, $x=3\sqrt{29}$.

When *CD* = 3. $x^2=3^2+(12+6)^2$, $x=3\sqrt{37}$.

Therefore, there are 6 possible values of *x*. Hence, we select option (E).

Answer: (E)

20. For any real number *x*, we denote by [*x*] the greatest integer not greater than *x*. For example, $[\pi]=3$ and $[-\pi]=-4$. How many positive integers *n* satisfy $\left[\frac{\left[\frac{100}{n}\right]}{n}\right]=1$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

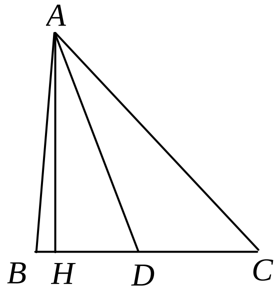
【Suggested Solution】

Since $\sqrt{100}=10$, we know that when the values of n increases then the value of $\left[\frac{\left[\frac{100}{n}\right]}{n}\right]$ will then decrease. Among the values of n as 11, 10, 9, 8, 7, 6 we discovered only 10, 9 and 8 satisfy the condition of the problem. Thus, there are 3 possible values of n. Hence, we select option (C).

Answer: (C)

【Remarks】When m, n are positive integers, we have $\left[\frac{\left[\frac{x}{m}\right]}{n}\right]=\left[\frac{x}{mn}\right]$, then the values of $\left[\frac{100}{n^2}\right]=1$, which is equivalent as $1\le\frac{100}{n^2}<2$, that is n = 8, 9, 10.

21. In the diagram, AH is perpendicular to BC, $AB = BC < AC$, and AD is the bisector $\angle BAC$. If $\angle DAH = 21°$, what is the measure, in degrees, of $\angle BAC$?



【Suggested Solution】

Let $\angle BAC=\alpha$.

Since AB = BC then $\angle C=\angle BAC=\alpha$,

Because AD is the angle bisector of BC, then $\angle DAC=\frac{1}{2}\angle BAC=\frac{\alpha}{2}$.

In right-angled triangle ACH, the two acute angles are complementary, then

$\alpha+\frac{\alpha}{2}+21°=90°$, so that $\alpha=46°$.

Answer: 046

22. How many four-digit numbers are divisible by all of 2, 3, 4, 5, 6, 7 and 8?

【Suggested Solution】

The Least Common Multiple of 2, 3, 4, 5, 6, 7 and 8 is 3 × 5 × 7 × 8 = 840, then

840 × 2, 840 × 3, 840 × 4, ⋯ ,840 × 11

We have a total of 10 four-digit numbers that satisfy the given condition.

Answer: 010

23. Each of A, B, C and D has some apples. A has as many apples as the other three together. B has half as many apples as the other three together. C has one-sixth as many apples as the other three together. How many times D's number of apples will be equal to total number of apples of A, B and C?

【Suggested Solution】

From the given information, A has $\frac{1}{1+1}=\frac{1}{2}$ the apples of all the four persons, B has $\frac{1}{1+2}=\frac{1}{3}$ the apples of all the four persons, C has $\frac{1}{1+6}=\frac{1}{7}$ the apples of all the four persons, then D has $1-\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{7}\right)=\frac{1}{42}$ the apples of all the four persons.

Therefore, the total number of apples that A, B and C have is $42-1=41$ times that D has.

Answer: 041

24. In how many ways can 31 be expressed in the form $a + b + c$ $(a\le b\le c)$, where a, b and c are prime numbers?

【Suggested Solution】

From the condition, $a + b + c = 31$ $(a\le b\le c)$. Since $3a\le 31$ then $a\le 10$, so the possible values of a are 2, 3, 5, 7.

If $a = 2$, then $b + c = 29$, no solutions!

If $a = 3$, then $b + c = 28$, there are 2 possible solutions: $(b, c) = (5, 23), (11, 17)$.

If $a = 5$, then $b + c = 26$, there are 2 possible solutions: $(b, c) = (7, 19), (13, 13)$.

If $a = 7$, then $b + c = 24$, there are 2 possible solutions: $(b, c) = (7, 17), (11, 13)$.

In summary, there are a total of 2 + 2 + 2 = 6 ways to express 31 as sum of three distinct prime numbers.

Answer: 006

25. Each of the three dimensions of a cuboid of volume a cm^3 is an integral number of cm. The cuboid is placed on a table. The total surface area of the five visible faces is a cm^2. Find the minimum value of a.

【Suggested Solution】

Let the length, width (or breadth) and height of the cuboid represent as x, y, z cm, respectively, then from the given information we have $xy + 2yz + 2zx = xyz$.

Divide both side of the above equation by xyz, $\frac{1}{z}+\frac{2}{x}+\frac{2}{y}=1$.

When $\frac{1}{z}=\frac{2}{x}=\frac{2}{y}=\frac{1}{3}$, then $x = y = 6$, $z = 3$, $xyz = 108$,

Suppose not all of $\frac{1}{z}$, $\frac{2}{x}$, $\frac{2}{y}$ is equal to $\frac{1}{3}$, then one of them greater than $\frac{1}{3}$.

Assume $\frac{1}{z}>\frac{1}{3}$, then $z < 3$, and because $\frac{1}{z}<1$ so that $z > 1$, then $z = 2$, so the given equation becomes $\frac{1}{x}+\frac{1}{y}=\frac{1}{4}$.

Thus, the possible $(x, y) = (5, 20), (6, 12), (8, 8), (12, 6), (20, 5)$. If follows the corresponding value of $xyz = 200, 144, 128$.

When $\frac{2}{x} > \frac{1}{3}$ (similarly for $\frac{2}{y} > \frac{1}{3}$) , then $x < 6$, and because $\frac{2}{x} < 1$ then $x > 2$, so that $x = 3, 4, 5$.

Suppose $x = 3$, we have $\frac{1}{z} + \frac{2}{y} = \frac{2}{3}$, the solution of $(y, z) = (7, 21), (8, 12), (9, 9), (12, 6), (15, 5), (24, 4)$. The corresponding product of $xyz = 441, 288, 243, 225, 216$.

Suppose $x = 4$, we have $\frac{1}{z} + \frac{2}{y} = \frac{1}{2}$, the solution of $(y, z) = (5, 10), (6, 6), (8, 4), (12, 3)$. The corresponding product of $xyz = 200, 144, 128$.

Suppose $x = 5$, we have $\frac{1}{z} + \frac{2}{y} = \frac{3}{5}$, then the solution of $(y, z) = (4, 10), (5, 5), (20, 2)$. The corresponding value of $xyz = 200, 125$.

In summary, the minimum volume of a is 108 cm^3, it is obtained when length is 6 cm, width is 6 cm and height is 3 cm.

Answer: 108

【Remarks】This problem is related with AM-GM Inequality.

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